832 Final exam: due May 17

(Take-home; Work out two problems independently, total 100 points.)

1. Running of coupling constants

The renormalization group equation (RGE) for a renormalized coupling g with respect to a physical (renormalization) scale μ is

$$\frac{dg(\mu)}{d\ln\mu} = \beta(g). \tag{1}$$

With n_f charged fermions in a gauge theory, the one-loop β functions are known to be

$$\beta(g) = \begin{cases} \frac{2n_f}{48\pi^2} g^3 & \text{for a U(1) Abelian gauge group,} \\ \\ \frac{2n_f - 11N}{48\pi^2} g^3 & \text{for an SU(N) Non – Abelian gauge group.} \end{cases}$$

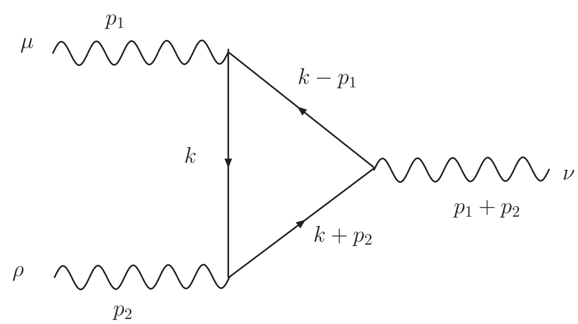
(a). Solve the RGE's and find the evolutions of the coupling constant versus the energy scale μ.

(b). Discuss the asymptotic behavior at high and low energies. Consider the two cases:

- The U(1) as electromagnetism $g = e,\ n_f = 1$,
- The SU(N) as QCD with $g = g_s,\ N = 3,\ n_f = 6$.

(c). It is customary to express the gauge couplings as $\alpha_N = g_N^2/4\pi$. If we take the approximate values of gauge couplings at the weak scale as $\alpha_1(100 \text{ GeV}) \approx 1/60$ and $\alpha_3(100 \text{ GeV}) \approx 1/8$, at what scale do these coupling strengths become about equal?

2. Trangle diagrams
Consider a trangle diagram of three external vector fields with fermions running in the loop, as shown



There is another contributing dragram like the above, but with the fermions running in the opposite direction (change the sign for all internal momenta, such as $k \leftrightarrow -k$).

(a). In QED with three external photons, write down the 1-loop amplitude $iT_{\mu\nu\rho}(p_1, p_2)$. Show that the contributions from the two diagrams cancel. (Hint: You do not need to calculate the integral to show the cancellation. You can make use of the Charge-conjugation operators acting on the γ matrices under the trace. Therefore, you can shown that this cancellation is true for any odd number of external photons. This is the Furry's Theorem.)

(b). Show that the divergent part of one of the diagrams is given by

$$iT_{\mu\nu\rho}(divergent) = -\frac{ie^3}{2\pi^2\epsilon}\left[\frac{1}{6}\left(g_{\mu\nu}(p_1-p_2)_\rho + g_{\mu\rho}(p_1-p_2)_\nu + g_{\rho\nu}(p_1-p_2)_\mu \right) + \frac{1}{2}\left(g_{\mu\nu}(p_1+p_2)_\rho + g_{\mu\rho}(p_1+p_2)_\nu + g_{\rho\nu}(p_1+p_2)_\mu \right)\right].$$

(It's kind of tedious – standard 1-loop calculations. See how far you can go.)