731 Homework set I (due Sept.16, 2005)

1. (10 points) Consider a one-dimensional harmonic oscillator with energy given by

$$E = \frac{p_x^2}{2m} + \frac{1}{2} m\omega^2 x^2.$$

Apply the Sommerfeld quantization condition $\oint p_x dx = nh$, where h is the Planck constant and $n = 0, 1, 2...$, derive the rule for energy quantization.

Hint: E is a constant. $\oint p_x dx = 2 \int_{-x_{max}}^{x_{max}} |p_x| dx$.

[Note that the Bohr and Sommerfeld quantization conditions are at best good guess to get something close to be right. They should not be viewed as fundamental principles.]

2. (10 points) For a system of particles of mass m in the state $\psi(\mathbf{r}, t)$, the particle density is defined as

$$\rho = \psi^* \psi,$$

and the flux (the number of particles per unit time through unit area perpendicular to the moving direction) is

$$\mathbf{J} = -i \frac{\hbar}{2m} (\psi^* \bigtriangledown \psi - \psi \bigtriangledown \psi^*).$$

Show that for a beam of free particles, the velocity can be expressed as

$$\mathbf{v} = \mathbf{J}/\rho.$$

You can make use of the free particle wave function

$$\psi(\mathbf{r}, t) = A \ \exp\{i(\mathbf{k} \cdot \mathbf{r} - \omega t)\},$$

with a velocity

$$\mathbf{v} = \frac{\hbar}{m} \mathbf{k}.$$

3. (20 points) A free particle is represented by the one-dimensional wave function

$$\psi(x,t) = A\ \exp\{i(kx - \omega t)\},$$

with $\hbar k = p$, and $\hbar\omega = E$. Connecting to the particle kinematics

$$\begin{aligned} E &= \frac{p^2}{2m}, \quad v = \frac{p}{m} \quad \text{for nonrelativistic case,} \\ E^2 &= p^2c^2 + m^2c^4, \quad \frac{v}{c^2} = \frac{p}{E} \quad \text{for relativistic case.} \end{aligned}$$

(1) Calculate the group velocity of the wave $u_g = d\omega/dk$, given in terms of the particle velocity v.

(2) Calculate the phase velocity of the wave $u_p = \omega/k$, given in terms of the particle velocity v.

(3) Comment on your results above by comparing the non-relativistic and relativistic cases.