731 Homework set 12 (due Dec. 9)

Density operator for $j = \frac{1}{2}$ system:

(a). Any 2×2 matrix M can be written in terms of the unit matrix I and the Pauli matrices σ_i as

$$M = \frac{1}{2}(a_0\ I + \vec{a} \cdot \vec{\sigma}), \tag{1}$$

where $a_0,\ a_i$ are constants. Show that $a_0 = Tr(M),\ \vec{a} = Tr(M\vec{\sigma})$. (5 points)

(b). Using the result in Eq. (1), show that the density operator for a spin-$\frac{1}{2}$ state $|\alpha\rangle$ can be expressed as (5 points)

$$\rho = \frac{1}{2}[I + \vec{P} \cdot \vec{\sigma}], \tag{2}$$

where $\vec{P} \equiv [\vec{\sigma}] = \langle\alpha|\vec{\sigma}|\alpha\rangle$ is the polarization vector under the state $|\alpha\rangle$.

(c). Assume that $|\alpha\rangle$ is the eigenstate for $\sigma_n = +1$, namely

$$|\alpha\rangle \doteq \begin{pmatrix} \cos\frac{\theta}{2}\ e^{-i\phi/2} \\ \sin\frac{\theta}{2}\ e^{i\phi/2} \end{pmatrix}, \quad \text{with} \quad \vec{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta),$$

show that under this state, the polarization vector $\vec{P} = \vec{n}$. (5 points)

(d). Construct the corresponding density matrix $\rho(\theta, \phi)$ under $|\alpha\rangle$ by definition, and verify the result by the explicit calculation from Eq. (2).
Does ρ represent a pure ensemble? (15 points)

(e). Now assume that the spin orientation $\vec{n}$ is random, find the angular averaged density matrix (10 points)

$$\rho_A = \int \rho(\theta, \phi)\ \frac{d\Omega}{4\pi}. \tag{3}$$

Does ρ_A represent a pure ensemble? Determine the polarization vector for this system. (10 points)