831 Homework set 3 (due Sept. 29)

Problem 1

The simplest non-cyclic group is of order 4. It is usually called the *four-group* or *dihedral group* denoted by D_2. If we denote the four elements by {e,a,b,c}, the multiplication table is given by

$$\begin{matrix} e & a & b & c \\ a & e & c & b \\ b & c & e & a \\ c & b & a & e. \end{matrix}$$

(1). Is D_2 an Abelian group? Justify your answer.

(2). What are the subgroups of D_2 if any?

(3). In the two-dimensional Euclidean space, consider a group of four elements: e as the identity, a the reflection in the horizontal direction, b the reflection in the virtical direction, c the rotation by π around the origin. Write down the 2×2 matrix representation of these operations on the basis $(\hat{x}, \hat{y})$, and justify that it forms a representation of D_2.

Problem 2

An alternative way of writing the Lie algebra for $SO(3)$ can be obtained by defining $J^{kl} = \epsilon^{klm} J_m,\ (k, l, m = 1, 2, 3)$ as generators for rotations in k-l plane. Show that

$$[J^{kl}, J^{mn}] = i(\delta^{km} J^{ln} - \delta^{kn} J^{lm} - \delta^{lm} J^{kn} + \delta^{ln} J^{km}).$$

Although this form seems to be less compact than the traditional relations, it is more readily generalized to higher dimensions.
Hint: The following contraction relations may be useful:

$$\epsilon_{\alpha\beta\gamma}\epsilon^{\alpha\beta\gamma} = 6, \quad \epsilon_{\alpha\beta\gamma}\epsilon^{\mu\beta\gamma} = 2\delta^{\mu}_{\alpha}, \quad \epsilon_{\alpha\beta\gamma}\epsilon^{\mu\nu\gamma} = \delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta},$$

$$\epsilon^{\alpha\beta\gamma}\epsilon_{ijk} = det(\delta^a_b) = \begin{vmatrix} \delta^{\alpha}_i & \delta^{\alpha}_j & \delta^{\alpha}_k \\ \delta^{\beta}_i & \delta^{\beta}_j & \delta^{\beta}_k \\ \delta^{\gamma}_i & \delta^{\gamma}_j & \delta^{\gamma}_k \end{vmatrix}, \qquad \text{where } a = \alpha, \beta, \gamma; \quad b = i, j, k.$$