

832 Homework Set 4 (due March 29)

1. Path integral for one-dimensional quantum mechanics

We have shown in class that the transition amplitude can be written in terms of the path integral

$$\langle q'', t'' | q', t' \rangle = \int \mathcal{D}p \mathcal{D}q \exp \left[i \int_{t'}^{t''} dt (p \dot{q} - H) \right] = \int \mathcal{D}q \exp \left[i \int_{t'}^{t''} dt L \right],$$

here and henceforth, we ignore overall constants (like $\sqrt{2\pi i \epsilon / m}$) that can be absorbed into the definition of $\mathcal{D}q$.

The correlation functions for the operators $Q(t)$, $P(t)$ are given by

$$\langle q'', t'' | TQ(t_i) \dots P(t_j) \dots | q', t' \rangle = \int \mathcal{D}p \mathcal{D}q q(t_i) \dots p(t_j) \dots \exp \left[i \int_{t'}^{t''} dt (p \dot{q} - H) \right].$$

(a). Functional derivatives:

We now introduce external forces f , h acting on the particle,

$$\langle q'', t'' | q', t' \rangle_{f,h} = \int \mathcal{D}p \mathcal{D}q \exp \left[i \int_{t'}^{t''} dt (p \dot{q} - H + fq + hp) \right].$$

Show that

$$\langle q'', t'' | TQ(t_i) \dots P(t_j) \dots | q', t' \rangle = \frac{\delta}{i\delta f(t_i)} \dots \frac{\delta}{i\delta h(t_j)} \dots \langle q'', t'' | q', t' \rangle_{f,h} |_{f,h=0}.$$

(b). Vacuum transition:

Show that

$$\langle 0 | 0 \rangle_{f,h} = \int \mathcal{D}p \mathcal{D}q \exp \left[i \int_{-\infty}^{\infty} dt (p \dot{q} - (1 + i\epsilon)H + fq + hp) \right].$$

[Hint: Consider $|q', t'\rangle = e^{-iHt'} |q'\rangle$. Expand in terms of energy eigenstates $|n\rangle$ and take $t' \rightarrow \infty$.]

2. Path integral for one-dimensional harmonic oscillator

Let the Hamiltonian be

$$H(P, Q) = \frac{1}{2}P^2 + \frac{1}{2}\omega^2 Q^2,$$

where we have set $m = 1$.

(a). Action in energy representation:

$$\langle 0|0\rangle_f = \int \mathcal{D}p\mathcal{D}q \exp \left[i \int_{-\infty}^{\infty} dt (p \dot{q} - (1 - i\epsilon)H + fq) \right] = \int \mathcal{D}q e^{iS},$$

with

$$S = \int_{-\infty}^{\infty} dt \left(\frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2 + fq \right).$$

Perform Fourier transformation,

$$q(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt} \tilde{q}(E),$$

and for $f(t)$ as well, express the action S by an energy integration.

(b). Show that

$$\langle 0|0\rangle_f = \exp \left[-\frac{i}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{\tilde{f}(E)\tilde{f}(-E)}{E^2 - \omega^2} \right].$$

[Hint: Making the action into a perfect quadratic form by changing the integration variables $\tilde{q}(E) = \tilde{x}(E) - \tilde{f}(E)/(E^2 - \omega^2)$. Also, you need to identify that $\int \mathcal{D}x \exp \left[\frac{i}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \tilde{x}(E)(E^2 - \omega^2)\tilde{x}(-E) \right] = \langle 0|0\rangle_f|_{f=0} \equiv 1$.]

(c). Define the Green's function $G(t - t')$ as

$$\left(\frac{\partial^2}{\partial t^2} + \omega^2 \right) G(t - t') = \delta(t - t').$$

Verify that the solution is

$$G(t - t') = - \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iE(t-t')}}{E^2 - \omega^2},$$

and thus obtain

$$\langle 0|0\rangle_f = \exp \left[\frac{i}{2} \int_{-\infty}^{\infty} dt dt' f(t)G(t - t')f(t') \right].$$

(d). Using the functional derivatives, calculate the vacuum correlation functions

$$\langle 0|TQ(t_1)Q(t_2)|0\rangle, \quad \langle 0|TQ(t_1)Q(t_2)Q(t_3)Q(t_4)|0\rangle.$$