832 Homework Set 5 (due April 17)

Supersymmetric harmonic oscillator

A. Bose case: a warmup
Let the system be given by the Lagrangian

$$L_B = \frac{1}{2}\dot{q}^2 - \frac{1}{2}\omega^2 q^2.$$

where we have set $m = 1$.

(1). Using the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0,$$

derive the equation of motion for q and write down the classical solution $q(t)$.

(2). Define the conjugate momentum

$$p = \frac{\partial L}{\partial \dot{q}},$$

and impose the quantization relation

$$[q, p] = i,$$

derive the commutation relation for the creation and annihilation operators.

(3). Define the system Hamiltonian

$$H_B = \dot{q}\, p - L_B,$$

give H_B in terms of the creation and annihilation operators.

B. Fermi case: Grassmann variables
With a complex Grassmann variable ψ, it is convenient to take the two independent variables as the pair $(\psi,\ i\psi^\dagger)$. Define the system by the Lagrangian

$$L_F = \frac{i}{2}(\psi^\dagger \dot{\psi} - \dot{\psi}^\dagger \psi) - \frac{\omega}{2}(\psi^\dagger \psi - \psi \psi^\dagger).$$

(1). Using the Lagrange equation, derive the equation of motion for ψ and show that the solutions are

$$\psi(t) = be^{-i\omega t}, \quad i\psi^\dagger(t) = ib^\dagger e^{i\omega t}.$$

(2). Impose the quantization relation $\{\psi, i\psi^\dagger\} = i$, derive the commutation relation for the creation and annihilation operators.

(3). Determine the system Hamiltonian H_F in terms of the creation and annihilation operators.

C. Supersymmetry

(1). With both bosonic and fermionic variables, write down the full Lagrangian and the Hamiltonian (again in terms of the creation and annihilation operators).

(2). Show that the action of the system $S = \int_{-\infty}^{\infty} dt L$ is invariant (upto total derivatives) under the following infinitesimal transformation

$$\delta q = \epsilon\psi + \psi^\dagger\bar{\epsilon}, \quad \delta\psi = (-i\dot{q} - \omega q)\bar{\epsilon}, \quad \delta\psi^\dagger = (i\dot{q} - \omega q)\epsilon,$$

where $\epsilon,\ \bar{\epsilon}$ are Grassmann numbers (constants).

(3). Introduce a pair of conjugate fermionic operators $(Q,\ \bar{Q})$,

$$Q = (i\dot{q} - \omega q)\psi = -\sqrt{2\omega}\ a^\dagger b, \quad \bar{Q} = (-i\dot{q} - \omega q)\psi^\dagger = -\sqrt{2\omega}\ ab^\dagger.$$

Show that the infinitesimal transformation can be rewritten as

$$\delta q = [\bar{\epsilon}\bar{Q} + \epsilon Q, q], \quad \delta\psi = [\bar{\epsilon}\bar{Q}, \psi], \quad \delta\psi^\dagger = [\epsilon Q, \psi^\dagger].$$

Also, the charge corresponding to Q (called the supercharge) is conserved

$$\dot{Q} = i[H, Q] = 0.$$

(4). Show that the Hamiltonian can be expressed as $H = \{Q, \bar{Q}\}/2$, and that the energy eigenvalues $E_n \geq 0$, where the equal sign is necessary for the ground state with an unbroken supersymmetry $Q|0\rangle = 0$.