835 Homework set V (due Nov. 19, or ...)

You need to work out one of the three problems to get full credits. You should choose the suitably challenging ones for your own sake. You are of course encouraged to work out as much as you can.

Level I Problems:

(1) Consider the hadronic production of the $W^{\pm}$ gauge bosons via the Drell-Yan mechanism

$$pp,\ p\bar{p} \to W^{\pm} + X, \tag{1}$$

where X denotes the unspecified inclusive final state. Write down the all combinations of the contributing partons for both pp and $p\bar{p}$ colliders, and for both W^+ and W^- production, Indicate if the partons are the valence or the sea quarks. Include only the quarks in the first two generations, but with the appropriate CKM mixing.

(2) Tabulate or plot the rapidity and pseudo-rapidity as a function of the polar angle θ between $1 \le \theta^\circ \le 179^\circ$,

• y_w for $M_w = 80$ GeV and $E_w = 200$ GeV.

• η for a massless particle.

(Note: It is enough to make the rapidity interval about 0.5 or so. And you should memorize a few typical values of $\eta(\theta)$.)

(3) Consider a particle with a four-momentum $p^\mu = (E, p_x, p_y, p_z)$. Define the transverse momentum and the rapidity

$$p_T = \sqrt{p_x^2 + p_y^2}, \quad y = \frac{1}{2}\ln\frac{E + p_z}{E - p_z}, \tag{2}$$

respectively, show that

• A particle four-momentum can be rewritten as

$$p^\mu = (E_T \cosh y, p_T \sin\phi, p_T \cos\phi, E_T \sinh y), \quad E_T = \sqrt{p_T^2 + m^2}. \tag{3}$$

• The longitudinal phase space element dp_z/E is invariant under the Lorentz boost along the z-direction.

• The phase space element then can be expressed as

$$\frac{d^3\vec{p}}{E} = p_T dp_T d\phi\ \frac{dp_z}{E} = E_T dE_T d\phi\ dy. \tag{4}$$

Level II Problems:

(1) The same as (2) in Level I.

(2) The same as (3) in Level I.

(3) Tabulate or plot the following total cross sections numerically in e^+e^- collisions versus the c.m. energy $\sqrt{s}$,

$$\begin{aligned}
&e^-\ e^+ \to \gamma^*,\ Z^* \to \mu^+\mu^-, \quad 50\ \text{GeV} < \sqrt{s} < 1000\ \text{GeV},\\
&e^-\ e^+ \to W^+W^-, \quad 2M_W < \sqrt{s} < 1000\ \text{GeV},\\
&e^-\ e^+ \to ZZ. \quad 2M_Z < \sqrt{s} < 1000\ \text{GeV},\\
&e^-\ e^+ \to \gamma^*,\ Z^* \to t\bar{t}, \quad 2m_t < \sqrt{s} < 1000\ \text{GeV},\\
&e^-\ e^+ \to Z^* \to Zh, \quad M_Z + m_h(120\ \text{GeV}) < \sqrt{s} < 1000\ \text{GeV}.
\end{aligned}$$

(Note: For the calculations, you may use whatever techniques and packages available to you. You do not need to present too fine intervals in $\sqrt{s}$ when way above the threshold. You should remember some cross section values at typical energies like $\sqrt{s} = 200, 500, 1000$ GeV etc.

Level III Problems:

(1) Same as (3) in Level II.

(2) Electron-positron annihilation to massless quark pair in $n-2\epsilon$ dimensions

$$e^-(p_1) + e^+(p_2) \to \gamma^* \to q(p_3) + \bar{q}(p_4).$$

• In terms of the Mandelstam variables $s_{ij} = (p_i + p_j)^2,\ t_{ij} = (p_i - p_j)^2$, as well as a scattering angle θ,

$$p_3 = \frac{\sqrt{s_{12}}}{2}(1, 0, ...0, \sin\theta, \cos\theta), \quad p_4 = \frac{\sqrt{s_{12}}}{2}(1, 0, ...0, -\sin\theta, -\cos\theta),$$

show that the two-body cross section is given by

$$\begin{aligned}
d\sigma_2 &= \frac{1}{2s_{12}} \overline{\sum} |\mathcal{M}_2|^2\ dPS_2,\\
\overline{\sum} |\mathcal{M}_2|^2 &= 2N_C\ e^4 Q_q^2\ (\frac{t_{13}^2 + t_{23}^2}{s_{12}^2} - \epsilon), \quad N_C = 3,\\
dPS_2 &= \frac{4^\epsilon}{16\pi} \left(\frac{4\pi}{s_{12}}\right)^\epsilon \frac{1}{\Gamma(1-\epsilon)} \int_0^\pi \sin^{1-2\epsilon}\theta d\theta.
\end{aligned}$$

(3) QCD corrections from a *soft* gluon emission off the quark lines

$$e^-(p_1) + e^+(p_2) \to \gamma^* \to q(p_3) + \bar{q}(p_4) + g(p_5).$$

Write the three-body cross section as

$$d\sigma_3 = \frac{1}{2s_{12}} \overline{\sum} |\mathcal{M}_3|^2 \; dPS_3.$$

• Show that in the limit $p_5 \to 0$, the matrix element factorizes as

$$\mathcal{M}_3 \approx \mathcal{M}_3^{soft} = g_s \mu^\epsilon \; J^\nu(p_5) \epsilon_\nu(p_5) \; \mathcal{M}_2,$$

with

$$J^\nu(p_5) = (\frac{p_3^\nu}{p_3 \cdot p_5} - \frac{p_4^\nu}{p_4 \cdot p_5}) T^a,$$

where μ is the renormalization scale in the dimensional regularization (to keep the gauge coupling dimensionless), and T^a the color matrix associated with the gluon-$q\bar{q}$ vertex.

• Parameterize the gluon momentum by

$$p_5 = E_g(1, ..., \sin\theta_1 \sin\theta_2, \sin\theta_1 \cos\theta_2, \cos\theta_1),$$

the soft gluon emission implies that $0 \le E_g \le \delta_s \sqrt{s_{12}}/2$, with $\delta_s \ll 1$. Show that

$$\begin{aligned} dPS_3 &\approx dPS_3^{soft} = dPS_2 \; \frac{d^{n-1} p_5}{2E_g (2\pi)^{n-1}} \\ &= dPS_2 \left(\frac{4\pi}{s_{12}}\right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} \; \frac{1}{2(2\pi)^2} \; dS, \\ dS &= \frac{1}{\pi} \left(\frac{4}{s_{12}}\right)^{-\epsilon} \int_0^{\delta_s \sqrt{s_{12}}/2} dE_g E_g^{1-2\epsilon} \sin^{-2\epsilon}\theta_2 d\theta_2 \; \sin^{1-2\epsilon}\theta_1 d\theta_1. \end{aligned}$$

• Put the above together, show that

$$\int \frac{1}{s_{35} s_{45}} dS \approx \frac{1}{2s_{12}} \; \left(\frac{1}{\epsilon^2} - \frac{2}{\epsilon} \ln \delta_s + 2 \ln \delta_s^2\right),$$

and thus,

$$
\begin{aligned}
d\sigma_3^{soft} &= \frac{1}{2s_{12}}\overline{\sum}|\mathcal{M}_3^{soft}|^2 \; dPS_3^{soft} \\
&= d\sigma_2 \left[\frac{\alpha_s}{2\pi}\,\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left(\frac{4\pi\mu^2}{s_{12}}\right)^{\epsilon}\right]\,\left(\frac{A_2}{\epsilon^2}+\frac{A_1}{\epsilon}+A_0\right), \\
A_2 &= 2C_F, \quad A_1 = -4C_F\ln\delta_s, \quad A_0 = 4C_F\ln^2\delta_s \quad C_F = 4/3.
\end{aligned}
$$

(Note: The $1/\epsilon^2$, $1/\epsilon$ poles will be cancelled by the 1-loop diagrams. Thus, the QCD corrections at the given order in perturbation theory is free of infrared divergence.)