831 Homework set 7 (due Oct.27)

Majorana fermion and Supersymmetry
Peskin-Schroeder: Problem 3.5 on page 74.

Remarks:
(a). There is a typo in the supersymmetry transformations. The χ field transformation should read

$$\delta\chi = \epsilon F + \sigma \cdot \partial\phi \; \sigma^2 \epsilon^*,$$

where $\sigma \cdot \partial\phi = \sigma^\mu \partial_\mu \phi$.

Note that the components of the spinors (ϵ, χ) are Grassmann numbers (see next page) and anti-commute.

(c). It's OK if you can't finish this part. Or you can just study the case $W = g\phi^3/3$. Try it out within your time limit, and I'll see how far you can go.

Grassmann Algebra

Motivated by the anti-commuting nature of the spinor operators in QFT, one postulates the corresponding properties for some classical objects, called Grassmann elements (or Grassmann numbers) and Grassmann variables.

(1) Grassmann elements:
The Grassmann elements ϵ_i $(i = 1, 2, ..., n)$ anti-commute between two of them

$$\epsilon_i \epsilon_j = -\epsilon_j \epsilon_i, \quad \text{thus } \epsilon_i^2 = 0,$$

and commute with normal constant numbers

$$c\epsilon_i = \epsilon_i c.$$

These properties can be summarized in terms of the *"graded elements"* α_i as

$$\alpha_i \alpha_j = (-1)^{n_i n_j} \; \alpha_j \alpha_i, \quad \text{where } n_i = \begin{cases} 0 & \text{for } \alpha_i \text{ bosonic}, \\ 1 & \text{for } \alpha_i \text{ fermionic}. \end{cases}$$

(2) Grassmann matrices:
Matrices composed by Grassmann numbers are Grassmann matrices, that possess some unusual properties for the transpose and complex conjugate:

$$(AB)^T = -B^T A^T, \qquad (AB)^* = -A^* B^*.$$

While these might be unfamiliar, the Hermitian conjugate "†" behaves as usual:

$$(AB)^\dagger = [(AB)^T]^* = B^\dagger A^\dagger.$$

(3) Grassmann algebra:
For "conjugate anti-commuting" variables θ and $\bar{\theta}$, a Grassmann algebra is defined by

$$\{\theta, \bar{\theta}\} = 0, \quad \theta^2 = \bar{\theta}^2 = 0, \quad \bar{\bar{\theta}} = \theta.$$