731 Homework set 8 (due Nov. 11)

1. Thermodynamics of oscillators:

Consider a one-dimensional simple harmonic oscillator system in thermal equilibrium at a temperature T. The statistical probability of finding the system in the state with an energy E_n is

$$P(n) = \frac{e^{-\beta E_n}}{Z},$$

where $Z = \sum_n e^{-\beta E_n}$ is the partition function and $\beta = 1/kT$ with k the Boltzman constant.

(a). Show that the thermal average of the system's energy, $\overline{E} = \sum_n E_n P(n)$, is (5 points)

$$\overline{E} = -\frac{\partial}{\partial \beta} \ln Z.$$

(b). Assume the system to be a classical oscillator. The state sum becomes a phase space integral

$$\sum_n \to \int \int dx \; dp.$$

Show that

$$Z_{cl} = \frac{2\pi}{\omega\beta} \quad \text{and} \quad \overline{E}_{cl} = \frac{1}{\beta} = kT.$$

How does this compare with your expectation based on the "equipartition theorem"? (10 pts)

(c). For a quantum mechanical harmonic oscillator system, show that (10 pts)

$$Z_{qm} = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}} \quad \text{and} \quad \overline{E}_{qm} = \hbar\omega \left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1} \right).$$

Under what condition do we recover $\overline{E}_{qm} \to \overline{E}_{cl}$? Comment your finding with respect to the Bohr's "correspondence principle". (5 pts)

2. Squeezed coherent state:
Consider a linear combination of the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ of a simple harmonic oscillator,

$$\hat{b} = u\hat{a} + v\hat{a}^\dagger, \qquad \hat{b}^\dagger = u^*\hat{a}^\dagger + v^*\hat{a} \ ,$$

where u and v are complex numbers.

(a). Derive a condition on u and v for which $\hat{b}$ and $\hat{b}^\dagger$ satisfy the commutation relation (5 points)

$$[\hat{b}, \hat{b}^\dagger] = 1 \ .$$

(b). Let $|\Omega\rangle$ be the state defined by

$$\hat{b}|\Omega\rangle = 0.$$

For what values of u and v does $|\Omega\rangle$ reduce to the familiar ground state $|0\rangle$? (5 points)

(c). Establish the differential equation for the wave function $\langle x|\Omega\rangle$ in the x-representation. (10 points) (Recall what I did in the lectures.)

(d). Obtain the explicit solution for $\langle x|\Omega\rangle$ in the x-representation. (5 points) (Treat u, v as real numbers for simplicity.)
Define a characteristic length analogous to the standard case $x_0 = \sqrt{\hbar/m\omega}$, and try to understand where the term "squeezed" comes from. (5 points)