731 Homework set 9 (due Nov. 18)

1. Group theory:

Let G be the group of continuous rotations in a plane around the origin, $G = \{R(\phi),\ 0 \le \phi \le 2\pi\}$. Let $\mathcal{V}^2$ be the corresponding 2-dimensional linear vector space.

(a). For an Euclidean vector $\mathbf{V} = (x_1,\ x_2)$ in $\mathcal{V}^2$, the rotation can be expressed as

$$\mathbf{V}' = D(\phi)\mathbf{V}, \quad \text{or} \quad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Show that $D(\phi)$ forms a 2-dimensional representation of the rotational group G. [Explicitly verify the group properties one by one.] (30 points)

(b). Show that the 2-dimensional representation $D(\phi)$ can be decomposed into two independent 1-dimensional representations. [Consider to choose the vectors in $\mathcal{V}^2$ to be complex, say $x_\pm = (x_1 \pm i x_2)/\sqrt{2}$.] (15 points)

(c). In the representation in (b), what is the form of the group generator? (5 points)

2. Consider a 2×2 matrix given by $\vec{\sigma} \cdot \vec{a}$, where $\vec{\sigma}$ is the Pauli matrices and $\vec{a}$ is an arbitrary vector in the 3-dimensional coordinate space. Perform a transformation

$$\vec{\sigma} \cdot \vec{a} \to \vec{\sigma} \cdot \vec{a}' = \exp(i\frac{\sigma_z}{2}\phi)\ \vec{\sigma} \cdot \vec{a}\ \exp(-i\frac{\sigma_z}{2}\phi). \tag{1}$$

(a). Find the explicit relation between $\vec{a}'$ and $\vec{a}$ in terms of ϕ. Comment on the physical meaning of the transformation. (20 points)

(b). Show that the transformation in Eq. (1) leaves the determinant and the trace of the matrix $\vec{\sigma} \cdot \vec{a}$ invariant. (10 points)